Analysis of wave propagation in a plate resting on a Winkler foundation with a sample application to vibration control

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A B S T R A C T

A variational statement in the field of complex numbers and its semi-analytical solution are presented to study the wave propagation in plates resting on a Winkler foundation. The method and its computer program are verified by comparison with the available results for a freely supported plate and performing a time domain analysis for a plate-Winkler foundation system. A variety of wave propagation phenomena is observed, and useful information can be extracted from the results. The foundation is found to act as a low frequency band-stop filter for flexural modes, and a stiffer foundation leads to wider widths of these stop bands. As a sample application, a barrier design for controlling flexural motions in the system is proposed. The effectiveness of the design is also demonstrated.

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1. Introduction

There are many soil models in the field of civil engineering, but the one used most frequently in soil–structure interaction analyses is the Winkler theory of equivalent springs because of its apparent simplicity and ability to describe many engineering problems and to “incorporate different nonlinear aspects of the behavior at a reduced computational effort compared to other approaches” [1]. The ground support of a plate in a working environment is usually modeled as equivalent springs according to the Winkler theory, as in the cases of ground floors of buildings, runways of airports and roadways. These plate- or beam-Winkler foundation systems have attracted much attention from many researchers for many years (e.g., [2–5]). However, none of these studies has revealed the problem of wave propagation, and they rely on plates of finite length. Furthermore, most of them have been done within the scope of classical or high-order shear deformation beam or plate theory.

Vibration is a major environmental concern through the world, and it is commonly described in terms of modes or wave motions. For a structure that behaves as if it has infinite length, using a wave model is more appropriate [6] because the modes of vibration are standing waves by nature and result from the interference of the traveling waves. In earthquake engineering, Rayleigh type waves propagating in the region near the soil surface transfer most of the vibration energy that may cause strong ground motions and stress levels that transmit the vibrations through the subsoil to the structures [7]. The plates on foundations are distinguished from the foundations by their waveguide geometry and, often, impedance differences. Mechanical vibrations in these plates, generated by seismic excitation, machine, traffic, etc., can be transmitted over long distances using a series of reflections between the cross section boundaries to the advantage of their propagation, which can cause distress to adjacent machines and structures and annoyance to people. Understanding the wave propagation information is important in studying near-field or high-frequency motions and developing an effective method of non-destructive evaluation, noise or vibration control, interpretation of earthquake records, etc. [8–17].

In this paper, a variational statement in the field of complex numbers is presented to study the problem of wave propagation in plates resting on a Winkler foundation. A semi-analytical technique is used to construct the variational trial function for the variational statement. The semi-analytical technique is based on the factorization of the function describing the displacement field of complex numbers. The cross section is modeled in a manner analogous to the conventional finite element method, while the displacement along the wave propagation direction is expressed analytically in the form of a harmonic complex exponential function. Available work relevant to a similar technique includes [18–20], in which other wave propagation problems are studied; [21,22], in which the buried structures subjected to three-dimensional surface loading are analyzed both statically and dynamically; and [23–25], in which the authors present the technique for the analysis of piezoelectric structures under mechanical and electrical loads.
2. Variational statement in the field of complex numbers for wave propagation analysis

Consider the spatial configuration of a general deformable body defined by domain \( \Omega \subset \mathbb{R}^3 \), which is described by Cartesian coordinates \((x,y,z)\) with Dirichlet boundary conditions on a given displacement area \( \delta \Omega \) and hybrid boundary conditions on the Winkler ground supported area \( \Omega \). Let \( \mathbf{u} \) be the unit normal vector to \( \partial \Omega \) and \( \mathbf{u}, \mathbf{v}, \sigma \) and \( \epsilon \) be the displacement vectors, fourth-order elastic, second-order stress and strain tensors, respectively, with their matrix forms:

\[
\begin{bmatrix}
\mathbf{u} \\
\mathbf{v} \\
\sigma
\end{bmatrix} = \begin{bmatrix}
\epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\
2\epsilon_{yz} & 2\epsilon_{zx} & 2\epsilon_{zy} \\
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} & \sigma_{yy} & \sigma_{yx} & \sigma_{yz} & \sigma_{zx} & \sigma_{zy}
\end{bmatrix},
\begin{bmatrix}
\mathbf{u} \\
\mathbf{v} \\
\sigma
\end{bmatrix} = \begin{bmatrix}
\mathbf{p} \\
\mathbf{q} \\
\mathbf{r}
\end{bmatrix},
\begin{bmatrix}
\mathbf{u} \\
\mathbf{v} \\
\sigma
\end{bmatrix} = \begin{bmatrix}
\mathbf{e} \\
\mathbf{C} \mathbf{e}
\end{bmatrix}
\]

for which the following relationships hold:

\[
\begin{align*}
\mathbf{u} & = \nabla [\mathbf{u}], \\
\nabla & = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z}
\end{bmatrix}, \\
\mathbf{e} & = \begin{bmatrix}
\epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\
2\epsilon_{yz} & 2\epsilon_{zx} & 2\epsilon_{zy} \\
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} & \sigma_{yy} & \sigma_{yx} & \sigma_{yz} & \sigma_{zx} & \sigma_{zy}
\end{bmatrix}, \\
\mathbf{C} & = \begin{bmatrix}
\mathbf{C} \\
\mathbf{C}
\end{bmatrix}
\end{align*}
\]

For harmonic waves with frequency \( \omega \), the common complex formalism should be adopted [26], and the displacement, strain, and stress are written as

\[
|\sigma, \epsilon, \mathbf{u}| = |\Psi| \mathbf{u}(x, y, z) \exp(-i\omega t)
\]

where \( i = \sqrt{-1} \) and \( \Psi, \epsilon, \mathbf{u} \) are the amplitudes. Thus, the kinetic energy of the structure \( \Omega \) is

\[
T = \frac{1}{2} \int_\Omega \rho \mathbf{u}^T \mathbf{u} \, d\Omega
\]

where \( * \) denotes the complex conjugate and \( \rho \) is the density. In the absence of body forces and external loads, the potential energy of the structure \( \Omega \) is

\[
V_p = \frac{1}{2} \int_\Omega \mathbf{V}_p \mathbf{u}^T \mathbf{u} \, d\Omega
\]

and the potential energy of the Winkler foundation is

\[
V_w = \frac{1}{2} \int_{\partial \Omega} k_z |\mathbf{u}| |\mathbf{n}| |\mathbf{n}|^T |\mathbf{u}| \, dS
\]

where \( k_z \) is the Winkler spring stiffness. Because Hamilton’s principle is in the form

\[
\int (T-V) \, dt = 0
\]

and \( T, V_p, V_w \) are independent of time \( t \) here, the variational statement is

\[
\delta \int (T-V) \, dt = 0
\]

where \( c.c. \) denotes the complex conjugate of the quantities that precede it. Eq. (8) is similar to the equations derived for harmonic motions in periodic structures in [26].

Let us consider a plate structure \( \Omega \) resting on the Winkler foundation and the harmonic waves propagating along the z-axis as shown in Fig. 1. For the Cartesian coordinates used to describe the system, the waves in the system are of the following plane wave form:

\[
|\mathbf{u}(x, y, z)\rangle = |\mathbf{u}(x, y, \omega, k_z)\rangle \exp(i k_z z)
\]

where \( \mathbf{u} \) is the amplitude and \( k_z \) is the wave number that describes the dispersive properties of the system and is defined in terms of the phase velocity \( c_p(\omega) \) and attenuation coefficient \( \gamma(\omega) \) [27]

\[
k_z = (\omega/c_p(\omega) + \omega/c_p(\omega))
\]

The wave number can be purely real, purely imaginary or complex, associated with a propagating, an evanescent or a decaying oscillating wave, respectively. If \( \omega(\omega(\omega)) \) is expanded in a Taylor series, the first-order expansion coefficient defines the group velocity [27]

\[
\frac{1}{c_g(\omega)} \equiv \frac{d \omega}{d c_p(\omega)} = \frac{1}{c_p(\omega)} \left( 1 - \frac{\omega}{c_p(\omega)} \right)
\]

It follows that

\[
\frac{d \omega}{d c_p(\omega)} = \frac{d c_z}{d k_z}
\]

Under normal dispersion conditions, which are in contrast to the abnormal conditions when the group velocity exceeds the phase velocity or becomes less than zero, the group velocity is the velocity of the wave package or the velocity at which energy is conveyed along the wave.

Substituting Eq. (9) into Eq. (8) and using the obvious facts that \( \exp(i k_z z) \exp(i k_z z) = 1 \) and the equation holds for the arbitrary z interval gives the following integration domain reduced equation:

\[
\hat{\delta} \hat{\mathbf{u}} = \int_{S_y} \nabla \delta \hat{\mathbf{u}} \cdot \mathbf{v} \, dS + \int_{T_o} -k_z \delta \hat{\mathbf{u}} \cdot \mathbf{n} \, dS + c.c. = 0
\]

where \( S_y \) is the cross section area of the plate, \( L_{ox} \) is the corresponding Winkler ground edge of \( S_y \) as shown in Fig. 1, and

\[
\nabla \mathbf{u} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
\nabla \mathbf{u} = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} \\
0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial z} & 0 & 0
\end{bmatrix}
\]
3. Semi-analytical finite element method solution

The plate cross section $S_W$ and its corresponding Winkler ground edge $L_0$ can be represented by a system of finite elements with domains $S$ and $L'$, respectively, as shown in Fig. 2, while the wave motion along the $z$-axis in Eq. (9) remains unchanged so that $z$-axis motion can be expressed analytically. In a typical element, the displacement field of complex numbers can be written in terms of the shape functions, $[N]^e$ and the nodal unknown displacements $[U]^e$; thus, the vector $[\hat{u}(x,y)]$ in Eq. (9) can be written as

$$[\hat{u}(x,y,o,k_2)]^e = [N(x,y)]^e[U(o,k_2)]^e \tag{15}$$

where the superscript $(^e)$ denotes an element level matrix or vector. The substitution of Eq. (15) into Eq. (13), followed by algebraic manipulations, yields

$$\delta [\hat{u}] = \sum_{i=1}^{ne} \int_S [\delta (U)^{ST} (\nabla_3 [N]^e)^T - ik_1 \nabla_3 [N]^e]^T [C]^e (\nabla_3 [N]^e)^T \] \, dx + \int_{L'} [\delta (U)^{ST} k_1 [N]^e] [N]^e [U]^e \, dz + c.c. = 0 \tag{16}$$

where $ne$ and $nef$ are the total numbers of elements in their corresponding domains. The number of elements and the order of the shape function should be consistent in the shared edge of the Winkler ground and plate cross section.

For the arbitrary variation of the indicated quantities $\delta [U]^{ST}$, this equation yields the following homogenous equation with each term at its global representation:

$$\sum_{j=1}^{ne} [K]^j [U]^f + \sum_{j=1}^{nef} [K]^j f [U]^f - \omega^2 \sum_{j=1}^{ne} [M]^j [U]^f = 0$$

$$[K]^f = \int_S [\delta (U)^{ST} [C]^e [B] \, dx \, dy \] \quad [M]^f = \int_S \rho [N]^e [U]^e \, dx \, dy \tag{17}$$

By defining the set $(\xi, \eta) \in \mathbb{R}^2 \, | \, (-1, 1)$ for an isoparametric element and mapping

$$f_1 : \mathbb{R} \rightarrow L' \subset \mathbb{R}^2$$
$$f_1 : (\xi, \eta) \rightarrow ([N(\xi, \eta)]^e [X]^f, [N(\xi, \eta)]^e [Y]^f) \in L'$$
$$f_2 : \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^2$$
$$f_2 : (\xi, \eta) \rightarrow ([N(\xi, \eta)]^e [X]^f, [N(\xi, \eta)]^e [Y]^f) \in S' \tag{18}$$

where $[X]^f, [Y]^f$ are the coordinates of element nodes, $[N(\xi, \eta)]^e$ is the same shape function as in Eq. (15), and $[N(\xi, \eta)]^e$ is the one dimensional standard shape function of the same order as $[N(\xi, \eta)]^e$, the entries in Eq. (17) can be expressed as

$$[K]^f = \int_{-1}^{+1} \int_{-1}^{+1} [B]^e [C]^e [B] \, dx \, dy$$
$$[M]^f = \int_{-1}^{+1} \rho [N]^e [U]^e \, dx \, dy$$
$$[K]^f = \int_{-1}^{+1} [k_1] [N]^e [n]^T \, dy \tag{19}$$

where $det[U(\xi, \eta)]$, $i = 1, 2$ is the determinant of Jacobian matrix $J(\xi, \eta)$ of mapping $f_1$ and $[\bullet]^e$ is the norm of vector $\bullet$. The normal vector $[n]^e$ in Eq. (19) can be obtained by mapping from the vector that is made by rotating a unit vector along the $\zeta$ direction in a parameter element by $90^\circ$.

$$[n] = [J]_{f_1(1)-1}, \quad R = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} \tag{20}$$

The standard finite element assembling procedures can be used here to rearrange Eq. (17) into the following generalized eigenvalue problem (EVP):

$$Ax = \lambda Bx \tag{21}$$

in which

$$x = \sum_{j=1}^{ne} [U]^f \, \lambda = \omega^2 \, \quad B = \sum_{j=1}^{ne} [M]^f$$

$$A = [K]^f + \sum_{j=1}^{ne} [K]^j f$$

where $x$ is the global vector of the unknown nodal displacements, $B$ is a real-valued symmetric matrix and $A$ is a Hermitian matrix. In the present case, all eigenvalues are real, and all eigenvectors satisfy orthogonality conditions [28].

Because only propagating waves are of interest in this paper, the EVP is solved for given values of real valued wave numbers $\Re(k_z)$. Thus, pairs $(\Re(k_z), \omega, \epsilon_p, c_p, c_s)$ and their eigenvectors give all of the propagating wave information in the system. However, the group velocity computed from its definition in Eq. (12) requires differentiation of discrete frequencies and wave numbers, and the discrete differentiation raises numerical challenges to obtain an accurate group velocity and convert the discrete solutions into different continuous branches of modes. This problem can be solved by following the approach used in [18–19]. First, the derivative of Eq. (21) with respect to the wave number is evaluated

$$\left( \frac{\partial A}{\partial k_z} - \frac{\partial \lambda}{\partial k_z} B \right) x + \left( A - \lambda B \right) \frac{\partial x}{\partial k_z} = 0$$

Alternatively, Eq. (21) can also be written in term of the left eigenvector $x^e$ as

$$x^e A = \lambda^e x^e B \tag{24}$$

and, for a complex Hermitian EVP, the following relations hold:

$$x^e = x^s \quad \lambda^e = \lambda \tag{25}$$

Then, multiply Eq. (23) from the left by the left eigenvector $x^e$, the following equation can be obtained:

$$\frac{\partial \lambda}{\partial k_z} = x^e \frac{\partial A}{\partial k_z} x^e / \lambda^e B x^e \tag{26}$$

Therefore,

$$c_g = \frac{\partial \lambda}{\partial k_z} = \frac{\partial \lambda^e}{\partial k_z} = x^e \frac{\partial A}{\partial k_z} x^e / 2 \lambda \lambda^e B x^e \tag{27}$$
In Eqs. (26) and (27),
\[
\frac{\partial A}{\partial k_z} = \sum_{j=1}^{\infty} \int_{s_0} \left(-i\mathbf{V}_d[N]^T[\mathbf{C}^*][\mathbf{V}_d[N]]^T ik_z \mathbf{V}_d[N]^T\right) + \sum_{j=1}^{\infty} \int_{s_0} \left(\mathbf{V}_d[N]^T - ik_z \mathbf{V}_d[N]^T\right)^T[\mathbf{C}^*](+i\mathbf{V}_d[N]^T)
\]
(28)

From Eq. (27), the group velocity can be solved without requiring the discrete differentiation.

In this paper, a C++ program based on the framework of OOFEM Lib [29] was written to form Eq. (21), and ARPACK++ [30], which is an object-oriented version of the well-known ARPACK [31] package, was used in the program to solve Eq. (21) efficiently. In the program, a second order 8-node 2D element (Fig. 2) for the plate cross section and the corresponding 3-node 1D element (Fig. 2) for the Winkler ground edge were developed in 3D space. They are general elements based on the continuous medium model and can catch all propagating modes in the plate-Winkler foundation system, provided the mesh of the cross section of the plate and Winkler foundation is sufficiently fine. It is an important part of developing an effective non-destructive evaluation and noise control method, where high frequency waves are of concern. When low frequency waves are of concern, the shell or plan strain assumption can be used in the foregoing derivation to reduce the cost of modeling.

4. Results and verifications

For an isotropic plate-Winkler foundation system, the advantage of symmetric geometry can be taken to de-couple symmetric motions from anti-symmetric motions, and only half of the cross section requires discretization, as shown in Fig. 2. For the sake of clarity, only symmetric motions are studied in this paper by enforcing \( U_y = 0 \) boundary conditions on the symmetry plane. The mesh refinement used in all of the examples is determined simply by convergence studies for the modes of interest, and the results are non-dimensionalized by the following expressions:

\[
\kappa = k_z H, \; \bar{\omega} = \frac{\omega H}{c_{ref}}, \; \bar{c}_g = \frac{c_g}{c_{ref}}, \; c_{ref} = \sqrt{\frac{E}{2\mu(1+\mu)}}
\]

Fig. 3. Non-dimensionalized typical results of propagating waves for the plate-Winkler foundation system: (a) frequency vs. wave number relationship, (b) group velocity vs. frequency relationship, and (c) enlarged views of the boxed areas around E11 and F21 modes.
A typical result is given first in what follows to describe certain common features of propagating waves for all cases studied. It also provides knowledge for later verification convenience. Next, a series of plate-Winkler systems with various foundation stiffnesses is considered to access the effect of the foundation. Finally, verification is conducted by two different processes.

4.1. Typical numerical results for the plate-Winkler foundation system

To show the typical results, a plate-Winkler system with the following geometry and physical property is considered:

\[ B = 5 \text{ m}, \quad H = 0.3 \text{ m}, \quad E = 3.1 \times 10^{10} \text{ N/m}^2, \]
\[ \rho = 23,000 \text{ N/m}^3, \quad \mu = 0.25, \quad k_f = 1.0 \times 10^{7} \text{ N/m}^3 \]

where \( E \) and \( \mu \) are Young’s elastic modulus and Poisson’s ratio of the isotropic plate, respectively.

Fig. 3 shows the wave number and group velocity for the lowest-order five positive direction waves. Various branches are classified according to their mode shapes and continuities. Special attention is paid around the frequencies where two dispersion curves approach each other closely. In these locations, wave veering may occur, where the two curves do not cross each other and the wave numbers change rapidly. The mode shape changes around the frequency where veering occurs [17]. Each branch or portion of the branch is denoted by a capital letter and two numbers, which indicate the predominant motion, branch number, and order of the mode shape, respectively. Predominant motions of flexure, extension, and expansion are denoted by the capital letters \( F, E, \) and \( X \), respectively.

Useful information on the propagating waves in the plate-Winkler system can be readily extracted from Fig. 3. For example,
it can be readily known that only three propagating modes, F11, F22 and E31, that propagate with group velocities of 268.7 m/s, 200.2 m/s and 1153 m/s, respectively, can be excited by an input of a 32 Hz (about 200 rad/s) external force to the system, as denoted by double arrow lines in Fig. 3(a) and (b). Their corresponding mode shapes are shown in Fig. 4. The 3D displacement field in Fig. 4 indicates that the E31 mode is the extensional mode characterized by its predominantly extensional motion; the F11 and F22 modes are the first and second order flexural modes, respectively, characterized by their predominantly bending motions.

In Fig. 3(b), phenomena of abnormal dispersion can be observed, for example, in the X51 and F21 modes (also see Fig. 3(c)). Wave veering is also observable, for example, in Fig. 3(a) and (b) between dispersion curves of branches 4 and 5 around where the changes of mode shape can be seen: the E41 mode stops to propagate around frequency 0.27 Hz and is converted to an expansion mode X41. Almost at the same location, a
new extensional mode E51 starts propagating, which is converted from an expansion wave X51. It is concluded from Fig. 3(b) that the fastest propagating waves are associated with those of extensional motions, which are relatively non-dispersive except around the locations of veering, and from Fig. 3(b) and (a), respectively, where a higher order flexural wave has a slower traveling speed and a longer wavelength along the propagating direction.

For comparison purposes, the dispersion relationship predicted by a corresponding classical plate on a Winkler foundation is briefly derived in Appendix A and solved numerically for several given wave numbers. The differences between the frequency results predicted by the classical theory of plates and this method is summarized in Table 1. The classical plate on a Winkler foundation can only predict the flexural modes, and, generally speaking, the prediction errors become larger as the wave number increases and the order of the wave mode increases, which is a consequence of the approximate nature of the classical theory used to describe the plate.

4.2. Numerical results for different Winkler foundation stiffnesses

One of the most important features of the guided wave is the cutoff frequency characterizing a boundary between a pass band and a stop band. In the work on vibration control and designing optical, electronic and acoustic devices, the material or structure is often designed to have desirable pass bands and stop bands using the cutoff features of the propagating waves in the system (e.g., [12–13]). To access the effects of foundation stiffness, hereafter, the cutoff frequencies for the waves shown in Fig. 3 are compared. The geometry and physical properties of the plate are the same as those in Eq. (30) but with foundations of various stiffnesses: $k_f=0.0–5.0 \times 10^9 \text{N/m}^3$. Fig. 5 shows the variation in cutoff frequencies as a function of foundation stiffness.

As expected, because the Winkler foundation is decoupled from the structure and takes only the displacements normal to the plate into account, only those propagating modes that have a displacement component in the normal direction are affected in Fig. 5, i.e., the flexural modes, and it is apparent that the plate with a stiffer Winkler foundation has higher cutoff frequencies for the flexural modes. Here, the Winkler foundation acts as a low frequency band-stop filter for the flexural modes, and a stiffer Winkler foundation leads to wider widths of these stop bands. In earthquake engineering, structural damage is often related to the linear amplification of seismic waves passing through soft soil layers near the earth’s surface [32,33], which is correct for small amplitudes of motion or small strains in the soil [32]. From the wave propagation point of view in this Winkler model of a foundation, this phenomenon can be explained by the passing effect of the soft site and filtering or suppressing effect of the stiffer site for the more destructive propagating flexural modes.

4.3. Verification by comparing with the available results for a freely supported plate

Published results exist for a freely supported thick plate in [20] (called a rectangular bar in the reference), so an important test is to check whether the program presented here can obtain the same results. Taken from [20], the plate has a Poisson’s ratio $\mu$ of 0.3 and a height to width ratio $H/B$ of 0.5. Because of the absence of a Winkler foundation and the double symmetric geometry of the plate cross section, only one-quarter of the cross section requires discretization. Four separate and uncoupled waveforms can be extracted: extension, torsion, and flexures about the x axis and y axis. The results obtained by this program are compared with those presented by Taweel et al. [20] in Fig. 6, where agreement with the published results can be seen.

4.4. Verification by performing a time domain analysis of a plate-Winkler foundation system

To verify the results in the time domain and to show how these results can be used to predict the time domain behavior of a physical plate-Winkler system, a time domain analysis was performed through a finite element simulation of the plate-Winkler foundation system studied in Section 4.1. As predicted in Section 4.1 and denoted in Fig. 3(a) and (b), only three modes, F11, F22 and E31, which propagate with group velocities of 268.7 m/s, 200.2 m/s and 1153 m/s, respectively, and have mode shapes shown in Fig. 4 can be excited by an input of 32 Hz. The goal of the analysis is to check whether the simulated responses are composed only of the three modes by an external excitation of 32 Hz and whether the three modes propagate at the velocities predicted by the results.
The commercial finite element program Abaqus/Explicit [34] was used for the simulation. The element used for the plate was C3D8, which is 8-node linear brick element, and the Winkler spring was SpringA, which is 2-node axial spring element. The plate-Winkler system has the same properties as in Eq. (30) but has a 400 m finite length. This long length facilitates the separation of wave modes from each other without interference from the end reflections. A schematic of the model is shown in Fig. 7. As shown in Fig. 7 (also refer to Fig. 1), the node in the upper left corner of the plate was chosen to be the excitation point. The external force was applied on the node in the x, y, and z directions simultaneously. The position and directions of the excitation were appropriate for representing a general vibration source and exciting most modes in the system. Fig. 8(a) shows the applied external force, which is a 9 circle sinusoidal wave of 32 Hz windowed by a Hanning window to reduce the spectral leakage caused by the truncation of the finite circles in the time domain. Its FFT transform is given in Fig. 8(b). As shown in Fig. 8(b), the real frequency content of the excitation has a range of about 10–50 Hz and is spread out over the desired frequency of 32 Hz. The displacement histories of nodes $L_i$, $M_i$ and $R_i$, $i=0–6$, as denoted in Fig. 7, were recorded during the analysis.

Consistent with the wave propagation model, no damping was assumed in the Abaqus analysis. The time incrementation scheme used was decided by Abaqus/Explicit automatically. Abaqus/Explicit uses an adaptive algorithm to determine conservative bounds for the highest element frequency. The mesh size in the wave propagation direction was chosen such that the propagating wave of interest with the shortest wavelength 4.44 m (F11 mode) was resolved spatially with at least 20 elements per wavelength. The element length used was set to 0.15 m. Obviously, a study of this wave propagation problem using the finite element method has a very high computational cost and is not so realistic in practice.

The results of the x, y, and z component displacements of the M5 node are shown as an example of results in Fig. 9(a–c), respectively. Apparently, there are only three modes excited. By referring to the 3D displacement fields shown in Fig. 4, the overwhelming wave packages of Mode F22 in Fig. 9(a), mode F11 and F22 in Fig. 9(b), and Mode E31 in Fig. 9(c) can be understood. The y-component waves of the $Li$ and $Ri$ nodes located on the two sides of plate are in anti-phase for the F22 mode (Fig. 4(c)) and in-phase for the F11 mode (Fig. 4(a)). These modes can be readily observable in Fig. 10, which shows the y-component displacement of the L5 and R5 nodes together.

Fig. 9. M5 node displacement history results of the time domain verification analysis: (a) x-component, (b) y-component, and (c) z-component.

Knowing the distance between two nodes, the group velocity of each wave package can be evaluated. Here, the peak of each wave package is used to accurately determine the time delay associated with the mode. Fig. 11 shows the y and z component displacements of the M4, M5 and M6 nodes together, from which the velocities were calculated as 266.5 m/s, 199.5 m/s and 1158.3 m/s for modes F11, F22 and E31, respectively. There is a good agreement with the velocities of 268.7 m/s, 200.2 m/s and 1153 m/s predicted by the results in Fig. 3(b).
5. Application example of a barrier design for vibration control

Vibration is a major environmental concern throughout the world. In the problem of a plate on a foundation, the flexural motions can cause large strains or radiation of noise in the sides of plate. Thus, from the wave propagation point of view, the goal of vibration control is to suppress the flexural modes. In this section, the same problem as the one studied in Section 4.4 is considered, and the propagating flexural modes, \( F_{11} \) and \( F_{22} \), excited by the 32 Hz frequency input, are to be controlled through a barrier design by applying the knowledge gained from the previous results.

Fig. 5 in Section 4.2 shows that it is possible to suppress the first and second order flexural modes at a frequency of 32 Hz by simply improving the foundation to a stiffer one, i.e., larger than \( k_f = \frac{0.4}{C^2} \times 10^9 \text{ N/m}^3 \). However, the full improvement of the foundation under the plate is not easy or even possible in practice. Thus, an alternative design is proposed by constructing a suitable wave barrier (local improvement of soil) in the path of the propagating waves, as shown in Fig. 12. As an example of an application, a designed foundation stiffness of \( k_f = \frac{1.0}{C^2} \times 10^9 \text{ N/m}^3 \) is chosen, and only one design variable \( d \) is considered. This proposed design has the advantage of being easy in practice and not causing discontinuity in the plate.

The propagating modes excited by the 32 Hz frequency input, \( F_{11} \) and \( F_{22} \), have wavelengths of \( \lambda_{F_{11}} = 4.44 \text{ m} \) and \( \lambda_{F_{22}} = 6.87 \text{ m} \), respectively, which can be calculated from their wave numbers, as shown in Fig. 3(a). The analysis was carried out for cases of \( d = 0.0, \ d = \frac{\lambda_{F_{11}}}{4}, \ d = \frac{\lambda_{F_{11}}}{2}, \ d = \frac{3\lambda_{F_{11}}}{4}, \ d = \lambda_{F_{11}}, \ d = 2\lambda_{F_{22}}, \ d = 2\lambda_{F_{22}}, \) and \( d = 220 \text{ m} \). Fig. 13 summarizes the maxima of the \( y \)-component displacement history of the nodes L5, M5 and R5 after the barrier for the different cases studied. Fig. 14(a–c) gives the typical \( y \)-component displacement history results of the nodes L5 after the barrier for cases \( d = 0.0, \ d = \frac{\lambda_{F_{11}}}{4} \) and \( d = 220 \text{ m} \), respectively.

The effectiveness of the design is clearly seen in Fig. 13. A barrier length of \( \frac{\lambda_{F_{11}}}{4} \) is sufficient to obtain a satisfactorily reduction result. A barrier length longer than \( \lambda_{F_{22}} \) makes the extensional mode \( E_{31} \) dominant even in the \( y \)-component history of the nodes after the barrier. Fig. 14(c) shows this effect clearly. Because of this dominant behavior of the \( E_{31} \) mode and the effect the design has only on flexural modes, a further barrier length increase would not cause a further reduction of the amplitude.

The effective vibration reduction design requires the knowledge of wave propagation in the system because the reduction effects depend on the frequency of vibration source, which can also be found in many real cases and papers in the literature, e.g., [7], where the authors conducted field experiment studies using

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**Fig. 11.** M4, M5 and M6 nodes displacement history results of the time domain verification analysis: (a) \( y \)-component, and (b) \( z \)-component.

**Fig. 12.** Schematic of the wave barrier design for the vibration control.

**Fig. 13.** Maximum \( y \)-component displacements of nodes L5, M5 and R5 after the barrier for different cases; \( d \) is non-dimensionalized by wavelength \( \lambda_{F_{11}} \), while the displacement by the maximum \( y \)-component displacement for case \( d = 0.0 \).

**Calculated velocity of each wave packet:**

- **F11 wave package:** 266.5 m/s.
- **F22 wave package:** 199.5 m/s.
- **E31 wave package:** 1158.3 m/s.
open and in-filled trench barriers to reduce the impact of soil vibrations on the structural response for both active and passive isolation cases. However, for the plate-foundation system, the reduction measures in [7] may not work because the waves can propagate with more vibration energy through the supported plate as well.

Similar to many engineering design practices, the application example concerns the selection of a Winkler soil parameter using the cutoff feature of the propagating wave. More features of the propagating waves can be exploited for practical uses, one good example of which is in [11], where much information on waves excitable by a high-speed train viaduct and in the bellowing soil are extracted from the dispersion curves. Then, a honeycomb wave impeding barrier is designed theoretically to modulate the waves and prevent undesirable waves from propagating through the soil into nearby buildings.

6. Summary and conclusions

This paper presents a variational statement in the field of complex numbers and its semi-analytical solution to study the problem of wave propagation in plates resting on a Winkler foundation, in which the continuous medium model is used because it has the ability to catch all propagating modes in the system. The method and its computer program are verified by (1) comparing with the available result for a freely supported thick plate and (2) performing a time domain analysis for a physical plate-Winkler foundation system. The latter also serves as an example to show how the results can be used to predict the time domain behavior of a physical plate resting on a Winkler foundation.

The results are given for symmetric motions and in the forms of wave number and group velocity for the five lowest-order positive direction waves in the frequency range studied. A variety of wave propagation phenomena is observed, and useful information about the propagating waves in the system can be readily extracted from the results. In parametrical studies, the foundation is found to act as a low frequency band-stop filter for flexural modes, and a stiffer foundation leads to wider widths of these stop bands. As an application example, a barrier design for controlling flexural motions in the plate-Winkler foundation system is proposed, and its effectiveness is also demonstrated. The effectiveness of controlling flexural motion is attributed to the inherent properties of the propagating wave in the system, and no material damping contributes.

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Appendix A

Let \( w(x,y,t) \) be the transverse displacement of the plate. Taking the resistive force of the Winkler foundation into account and following the derivation of classical plate (e.g., [35]), the following equation for the case of a classical plate resting on Winkler foundation can be obtained:

\[
D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + \rho h \frac{\partial^2 w}{\partial t^2} + k_f w = 0
\]  

(A.1)

where \( \rho \) is the density, \( h \) is the plate thickness, \( D = Eh^3/12(1 - \nu^2) \) and \( k_f \) is the Winkler stiffness. Consistent with the examples studied in this paper, only half of the plate-foundation system is modeled using the symmetry condition about the \( y=0 \) middle plane, and, to extract the symmetric motions propagating in the \( x \) direction, the system is assumed to have a symmetry boundary condition along the \( y=0 \) edge in addition to the free edge along

Fig. 14. \( y \)-component displacement histories of node L5 after the barrier: (a) case \( d=0.0 \) m, (b) case \( d=1.11/4=1.11 \) m, and (c) case \( d=220.0 \) m.
For harmonic waves with frequency $\omega$, waves have the following plane wave form:

$$ w = \tilde{w}(y, \omega t) \exp(i k x - \omega t) $$

(A.3)

where $\tilde{w}$ is the complex amplitude and $k$ is the complex wave number. By substituting Eq. (A.3) into Eq. (A.1), the following fourth order differential equation governing the waves is found

$$ D \left( \frac{d^4 \tilde{w}}{dy^4} - 2k^4 \frac{d^2 \tilde{w}}{dy^2} + \text{w}k^2 \right) - \rho \omega^2 \tilde{w} + k^4 \tilde{w} = 0 $$

(A.4)

The solution of the differential equation requires careful attention to boundary equations. Remarkable difficulties can arise in the statement of boundary conditions, and finding the analytical solution, even with the very simple boundary conditions, is rather complex or not possible in the present case. For only certain special cases, as with the usual solution of differential equations, a solution can be assumed that satisfies the boundary equations beforehand, e.g., Navier trigonometric series for the case of simply supported boundaries. The general solution to Eq. (A.4) is

$$ \tilde{w}(y) = \tilde{w}_i \exp(i \gamma_1 y) + \tilde{w}_2 \exp(i \gamma_2 y) + \tilde{w}_3 \exp(i \gamma_3 y) + \tilde{w}_4 \exp(i \gamma_4 y) $$

(A.5)

Forcing the solution Eq. (A.5) to fit the physical boundary conditions yields the following equation:

$$ \begin{align*}
|A_1| \sum_{i=1}^{4} |\tilde{w}_i| &= 0, \quad \text{and} \quad |A| = \\
&= \left\{ \\
&\quad \int_{0}^{2\pi} -ir_1 -ir_2 -ir_3 -ir_4 -i(2\mu - k^2) -k - r_1 & r_2 & r_3 & r_4 \\
&\quad \int_{0}^{2\pi} -ir_1 -ir_2 -ir_3 -ir_4 -i(2\mu - k^2) & r_1 & r_2 & r_3 & r_4 \\
&\quad \int_{0}^{2\pi} -ir_1 -ir_2 -ir_3 -ir_4 & r_1 & r_2 & r_3 & r_4 \\
&\quad \int_{0}^{2\pi} -ir_1 -ir_2 -ir_3 & r_1 & r_2 & r_3 & r_4 \\
&\quad \int_{0}^{2\pi} -ir_1 & r_1 & r_2 & r_3 & r_4 \\
\end{align*} $$

(A.7)

The determinant of the coefficients must vanish for nontrivial solutions of $\tilde{w}$ to exist, which gives the equation, i.e., $|A| = 0$, that describes the dispersion relationship (complex $k$ as a function of real $\omega$). It is impossible to solve the dispersion equation explicitly for $k$ in terms of $\omega$ or $\omega$ in terms of $k$, and when solving the dispersion equation numerically, only the real solutions of the equation, i.e., real $k$, are interesting because they represent the propagating modes of the system.

References


